Journal of Approximation Theory **97**, 220–239 (1999)
Article ID jath.1996.3272, available online at http://www.idealibrary.com on IDEAL®

Smoothness Theorems for Erdős Weights, II

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Communicated by András Kroó

Received October 11, 1996; accepted in revised form April 11, 1998

We obtain new characterizations of smoothness, saturation results, and existence theorems of derivatives for weighted polynomials associated with Erdős weights on the real line. Our methods rely heavily on realization functionals. © 1999 Academic Press

Key Words: Erdős weight; Jackson–Bernstein theorem; modulus of smoothness; realization functional; polynomial approximation.

1. INTRODUCTION

Recently, there has been much interest in the study of rates of polynomial approximation in weighted $L_p(0<p\leqslant\infty)$ spaces, associated with fast decaying weights on the real line and $[-1,1]$. We refer the reader to [1, 4, 7] and the references cited therein, for a detailed and comprehensive account of the above topic.

In this paper, we consider smoothness theorems in $L_p(0<p\leqslant\infty)$ for weighted polynomials associated with Erdős weights on the real line complementing earlier work of [1, 2], and [4]. In order to state our results, we need to define our class of weight functions and various quantities. First we say that a real valued function $f\colon(a,b)\to(0,\infty)$ is *quasi-increasing* if there exists a positive constant C such that

$$a<x<y<b\Rightarrow f(x)\leqslant Cf(y).$$

Our weight class will be assumed to be admissible in the sense of the following definition.

* The research of the author was completed while visiting the University of South Florida during the Fall Semester, 1996.

0021-9045/99 $30.00

DEFINITION 1.1. Let

$$W = \exp(-Q),$$

where $Q: \mathbb{R} \to \mathbb{R}$ is even and continuous. Then W is an admissible weight and we shall write $W \in \mathscr{E}$ if the following conditions below hold.

(a) $xQ'(x)$ is strictly increasing in $(0, \infty)$ with

$$\lim_{|x| \to 0^+} xQ'(x) = 0.$$

(b)

$$T(x) := \frac{xQ'(x)}{Q(x)}$$

is quasi-increasing in (C, ∞) for some $C > 0$ and

$$\lim_{|x| \to \infty} \frac{xQ'(x)}{Q(x)} = \infty.$$

(c) Assume that for each $\varepsilon > 0$, there exists $C_j > 0$, $j = 1, 2$ such that

$$\frac{yQ'(y)}{xQ'(x)} \leqslant C_1 \left(\frac{Q(y)}{Q(x)}\right)^{1+\varepsilon}, \qquad y \geqslant x \geqslant C_2. \tag{1.1}$$

It is instructive to present two classical examples of our admissible weights below.

(a)

$$W_{k,\alpha}(x) := \exp(-\exp_k(|x|^\alpha)), \qquad \alpha > 1, \qquad k \geqslant 1, \qquad x \in \mathbb{R}. \tag{1.2}$$

Here $\exp_k(;) := \exp(\exp(\cdots(\exp(;)))$ denotes the kth iterated exponential.

(b)

$$W_{A,B}(x) := \exp(-\exp(\log(A + x^2)^B)), \qquad x \in \mathbb{R}. \tag{1.3}$$

Here $B > 1$ and A is a fixed but large enough real number.

Armed with the above class of admissible weights above, we now define a suitable measure of weighted distance.

Let $I \subseteq \mathbb{R}$ be an interval and

$$L_{p,W}(I) := \{f : I \to \mathbb{R} : fW \in L_p(I), 0 < p \leqslant \infty\},$$

where if $p=\infty$, f is further continuous and satisfies

$$\lim_{|x| \to \infty} fW(x)=0.$$

We equip $L_{p,W}(I)$ with the quasi-norm

$$\begin{cases} \left(\int_I |fW|^p(x)\,dx\right)^{1/p}, & 0<p<\infty \\ \sup_{x \in I} |fW|(x), & p=\infty \end{cases}$$

and interpret $(L_{p,W}(I), \|\,;\,\|)$ as a metric space in the usual way. In particular, taking $I=\mathbb{R}$, we may define the $L_p(0<p\leqslant\infty)$ error in best weighted polynomial approximation by

$$E_n[f]_{W,p} := \inf_{P \in \mathcal{P}_n} \|(f-P)\,W\|_{L_p(\mathbb{R})}, \qquad f \in L_{p,W}(\mathbb{R}), \tag{1.4}$$

where $\mathcal{P}_n$ denotes the class of polynomials of degree at most $n \geqslant 1$.

In [1] and [4], Jackson and Bernstein estimates for $E_n[f]$ for fixed $f \in L_{p,W}(0<p\leqslant\infty)$ were investigated. In order to describe these results, we need the notion of the Mhaskar–Rakhmanov–Saff number and a suitable weighted modulus of smoothness which we define below.

Mhaskar–Rakhmanov–Saff Number

Let $W \in \mathcal{E}$ and define the Mhaskar–Rakhmanov–Saff number, a_u, $u \geqslant 0$, by the equation:

$$u=\frac{2}{\pi}\int_0^1 \frac{a_u t Q'(a_u t)}{\sqrt{1-t^2}}\,dt, \qquad u>0.$$

Then under our assumptions on Q, it was shown in [4] that a_u is uniquely defined and is a strictly increasing function of u. Moreover, it is continuous for $u \in (0,\infty)$ and satisfies for every fixed $\delta>0$

$$\frac{a_u}{u^\delta} \to 0, \qquad u \to \infty. \tag{1.5}$$

The Weighted Jackson Modulus of Continuity

The following weighted Jackson modulus of continuity was introduced and studied in [1, 2], and [4].

DEFINITION 1.2. Let $W \in \mathscr{E}$, $0 < p \leqslant \infty$, $f \in L_{p,W}(\mathbb{R})$, $r \geqslant 1$ and set

$$\omega_{r,p}(f, W, t) := \sup_{0 < h \leqslant t} \|\Delta^r_{h\Phi_t(x)}(f, x, \mathbb{R})\|_{L_p(|x| \leqslant \sigma(2t))}$$
$$= \inf_{R \in \mathscr{P}_{r-1}} \|(f - R) W\|_{L_p(|x| \geqslant \sigma(4t))}. \tag{1.6}$$

Here

(a)

$$\sigma(t) := \inf \left\{ a_u : \frac{a_u}{u} \leqslant t \right\}, \qquad t > 0. \tag{1.7}$$

(b)

$$\Phi_t(x) := \left| 1 - \frac{|x|}{\sigma(t)} \right|^{1/2} + T(\sigma(t))^{-1/2}, \qquad x \in \mathbb{R}. \tag{1.8}$$

For a real interval J,

$$\Delta^r_h(f, x, J) := \begin{cases} \sum\limits_{i=0}^{r} (r/i)(-1)^i f(x + (rh/2) - ih), & x \pm \dfrac{rh}{2} \in J \\ 0, & \text{otherwise} \end{cases}$$

is the rth symmetric difference of f.

The following remark assists in the assimilation of the complicated terminology above.

Remark 1.3. (a) The essential feature of the function σ in (1.7) is that it satisfies the following important condition. Uniformly for $n \geqslant 1$, there exist constants $C_j > 0$, $j = 1, 2$ independent of n such that

$$C_1 \leqslant \frac{\sigma(a_n/n)}{a_n} \leqslant C_2.$$

Thus, in a sense, $\sigma(a_n/n)$ serves as the inverse of the function

$$a_n : \to \frac{a_n}{n}, \qquad n \geqslant 1.$$

Typically, t is small and will be taken as a_n/n for $n \geqslant n_0$ for some fixed but large enough n_0.

(b) The function Φ_t is a suitable replacement for the well-known factor $\sqrt{1 - x^2}$ in the Ditzian–Totik modulus, i.e., it describes the improvement in the degree of approximation near $\pm a_{n/2}$.

(c) The tail of the modulus $\omega_{r,p}(f, W, ;\ )$ reflects the inability of (PW), $P \in \mathscr{P}_n$ to approximate beyond $[-a_{n/2}, a_{n/2}]$. Its presence ensures that for $f \in \mathscr{P}_{r-1}$, $r \geqslant 1$,

$$\omega_{r,p}(f, W, ;\ ) \equiv 0. \tag{1.9}$$

We finish this section with two important theorems which were established in [1] and [4]. In order to state them, we adopt the following convention that will be used in the sequel.

Throughout, for real sequences $\{A_n\}$ and $\{B_n\} \neq 0$,

> $A_n = O(B_n)$, $A_n \sim B_n$ and $A_n = o(B_n)$ will mean respectively that there exist constants $C_1, C_2, C_3 > 0$ independent of n such that $A_n/B_n \leqslant C_1$, $C_2 \leqslant A_n/B_n \leqslant C_3$ and $\lim_{n\to\infty} |A_n/B_n| = 0$.

Similar notation will be used for functions and sequences of functions.

THEOREM 1.4. *Let* $W \in \mathscr{E}$, $0 < p \leqslant \infty$, $f \in L_{p,W}(\mathbb{R})$, $r \geqslant 1$ *and* $n \geqslant n_0$. *Assume that there is a Markov–Bernstein inequality of the form*

$$\|R'\Phi_{a_n/n} W\|_{L_p(\mathbb{R})} \leqslant C_1 \frac{n}{a_n} \|RW\|_{L_p(\mathbb{R})}, \qquad R \in \mathscr{P}_n. \tag{1.10}$$

Then there exists $C_2 > 0$ *independent of* f *and* n *such that*

$$E_n[f]_{W,p} \leqslant C_2 w_{r,p}\left(f, W, \frac{a_n}{n}\right). \tag{1.11}$$

The result indicated a *Nikolskii–Timan–Brudnyi* effect whereby as in weights on $[-1, 1]$, we have better approximation towards the endpoints of the Mhaskar–Rakhmanov–Saff interval.

In order to establish (1.11), we used a natural realization functional defined by

$$K_{r,p}(f, W, t^r) := \inf_{P \in \mathscr{P}_n} \{\|(f-P) W\|_{L_p(\mathbb{R})} + t^r \|P^{(r)}\Phi_t^r W\|_{L_p(\mathbb{R})}\}. \tag{1.12}$$

Here $t > 0$ is chosen in advance and n depends on t by the following relation:

$$n = n(t) := \inf\left\{k : \frac{a_k}{k} \leqslant t\right\}. \tag{1.13}$$

The concept of realization should be attributed to Hristov and Ivanov [6]. It enabled us to use a general technique of Ditzian, Hristov, and Ivanov [6] to show

THEOREM 1.5. *Let* $W \in \mathcal{E}, 0 < p \leqslant \infty, f \in L_{p,W}(\mathbb{R}), r \geqslant 1, \alpha > 0$ *and assume* (1.10). *Let* $t \in (0, D)$ *where* D *is a small enough fixed positive number and determine* n *by* (1.13). *Then uniformly for* f *and* t *the following hold.*

(a)

$$\omega_{r,p}(f, W, t) \sim K_{r,p}(f, W, t^r). \tag{1.14}$$

(b)

$$\omega_{r,p}(f, W, t) \sim \omega_{r,p}(f, W, \alpha t) \sim \omega_{r,p}\left(f, W, \frac{a_n}{n}\right). \tag{1.15}$$

(c)

$$K_{r,p}(f, W, t^r) \sim \|(f - P_n^*) W\|_{L_p(\mathbb{R})} + t^r \|P_n^{*(r)} \Phi_t^r W\|_{L_p(\mathbb{R})}. \tag{1.16}$$

Here, $P_{n,p}^* = P_n^*$ *is the best aproximant to* f *from* $\mathcal{P}_n$ *satisfying*

$$\|(f - P_n^*) W\|_{L_p(\mathbb{R})} = E_n[f]_{W,p}. \tag{1.17}$$

(d) *Moreover if* $1 \leqslant p \leqslant \infty$ *and* f *satisfies the extra smoothness requirement*

$$f^r W \in L_p(\mathbb{R})$$

then there exists $C_1 > 0$ *independent of* t *and* f *such that*

$$\omega_{r,p}(f, W, t) \leqslant C_1 t^r \|f^{(r)} W\|_{L_p(\mathbb{R})}. \tag{1.18}$$

This paper is organized as follows: In Section 2, we present our main results. In Section 3, we establish Theorems 2.1 and 2.3. In Section 4, we present the proofs of Theorems 2.6, 2.7, 2.9, and 2.10.

2. STATEMENTS OF RESULTS

Throughout this paper, $C, C_1, ...$ will denote positive constants independent of t, n, x and $P \in \mathcal{P}_n$ while the symbol D will always denote a small enough but fixed positive constant. The same symbol does not necessarily denote the same constant in different occurrences. We shall write $C \neq C(L)$ to mean that the constant in question is independent of the parameter L.

2.1. *A Smoothness Inequality in L_p, $p \geqslant 1$*

In general, the constants in the $\sim$ relation in (1.15) depend on α and one has typically for the modulus $\omega^r(f, ;)_p$ of [8] the inequality

$$\omega^r(f, \lambda t)_p \leqslant C_1 \lambda^r \omega^r(f, t)_p$$

for $\lambda \geqslant 1$ and $p \geqslant 1$. Here $C_1 > 0$ is independent of f, t, and λ.

In this paper we prove

THEOREM 2.1. *Let $W \in \mathscr{E}$, $1 \leqslant p \leqslant \infty$, $f \in L_{p, W}(\mathbb{R})$, $r \geqslant 1$, and $t \in (0, D)$. Then uniformly for $\lambda \in [1, D/t]$, there exists $C_1 > 0$ independent of f and t such that*

$$w_{r, p}(f, W, \lambda t) \leqslant C_1 \lambda^r (\sup_{x \in \mathbb{R}} \Psi_{\lambda t, t}(x))^r \, w_{r, p}(f, W, t), \tag{2.1}$$

where for any $y, z > 0$

$$\Psi_{y, z}(x) := \frac{\Phi_y(x)}{\Phi_z(x)}, \qquad x \in \mathbb{R}. \tag{2.2}$$

In particular, given $\varepsilon > 0$, we have for $0 < t < D$ and uniformly for $\lambda \in [1, D/t]$,

$$w_{r, p}(f, W, \lambda t) \leqslant C_2 \lambda^{r+\varepsilon} w_{r, p}(f, W, t). \tag{2.3}$$

Here, C_2 is independent of t, f, and λ.

Remark 2.2. One can prove, under the hypotheses of Theroem 2.1 the following infinite-finite range inequality:

Let $\alpha > 1$, $\beta \in \mathbb{R}$ and $0 < t < D$. Define $n = n(t)$ by (1.13). Then for all $P \in \mathscr{P}_n$ and uniformly for $\lambda \geqslant 1$,

$$\|PW\Phi^{\beta}_{\lambda t}\|_{L_p(\mathbb{R})} \leqslant C_1 \, \|PW\Phi^{\beta}_{\lambda t}\|_{L_p(|x| \leqslant \sigma(t/4\alpha))}.$$

This enables us to replace

$$\sup_{x \in \mathbb{R}} \Psi_{\lambda t, t}(x)$$

in (2.1) by

$$\max_{|x| \leqslant \sigma(t/4\alpha)} \Psi_{\lambda t, t}(x).$$

However, as the proof of Lemma 3.2 will show, the main contribution of $\Psi_{\lambda t, t}(x)$ comes from the interval

$$\sigma\left(\frac{\lambda t}{4\alpha}\right) \leqslant |x| \leqslant \sigma\left(\frac{t}{4\alpha}\right)$$

so this replacement still yields (2.3) and is hardly worth the effort.

As a corollary, of the above, we are able to prove the following saturation type result complementing (1.9).

THEOREM 2.3. *Let* $W \in \mathscr{E}, 1 \leqslant p \leqslant \infty, f \in L_{p, W}(\mathbb{R})$ *and* $r \geqslant 1$. *Suppose that for a given* $\varepsilon > 0$,

$$\liminf_{t \to 0^+} \frac{\omega_{r, p}(f, W, t)}{t^{r+\varepsilon}} = 0. \tag{2.4}$$

Then f is a polynomial of degree $r-1$ a.e.

Remark 2.4. We observe that (2.4) is false for $0 < p < 1$.
Indeed set

$$f(x) := \begin{cases} 0, & x \in (-1, 0) \\ x^{r-1}, & x \in (0, 1). \end{cases}$$

Then $f \in L_p$, $p > 1$, f is of compact support and

$$\omega^r(f, t) := \sup_{0 < h \leqslant t} \|\Delta_h^r(f)\|_{L_p(-1, 1)} = O(t^{r-1+1/p}).$$

As f is of compact support,

$$\omega^r(f, t) \sim \omega_{r, p}(f, W, t).$$

It remains to observe that a polynomial of degree $r-1$ of compact support $\equiv 0$.

2.2. *A Characterization Theorem*

In order to formulate our next two results, we need the following characterization theorem which was proved in [1].

THEOREM 2.5. *Let* $W \in \mathscr{E}, 0 < \alpha < r, 0 < p \leqslant \infty, f \in L_{p, W}(\mathbb{R})$ *and assume* (1.10).

Then the following are equivalent.

(a)

$$E_n[f]_{W,p} = O\left(\frac{a_n}{n}\right)^{\alpha}, \qquad n \to \infty. \tag{2.5}$$

(b)

$$\omega_{r,p}(f, W, t) = O(t^{\alpha}), \qquad t \to 0^{+}. \tag{2.6}$$

Observe that Theorem 2.5 does not include the case $\alpha = r$. To this end, we replace (2.5) by a different characterization and prove:

THEOREM 2.6. *Let* $W \in \mathscr{E}$, $1 \leqslant p \leqslant \infty$, $f \in L_{p,W}(\mathbb{R})$ *and assume* (1.10). *Suppose further that*

$$\|P_n^{*(r)} \Phi_{a_n/n}^r W\|_{L_p(\mathbb{R})} \leqslant C_1 \left(\frac{n}{a_n}\right)^r \psi\left(\frac{a_n}{n}\right), \qquad n \to \infty \tag{2.7}$$

for some quasi-increasing

$$\psi: [0, \infty] \to [0, \infty]$$

satisfying

$$\psi(x) \to 0, \quad x \to 0^{+}.$$

Then,

(a)

$$E_n[f]_{W,p} \leqslant C_2 \left(\int_0^{C_3(a_n/n)} \frac{\psi(\tau)}{\tau}\, d\tau\right), \qquad n \to \infty \tag{2.8}$$

and

$$\omega_{r,p}(f, W, t) \leqslant C_4 \left(\int_0^{C_5 t} \frac{\psi(\tau)}{\tau}\, d\tau\right), \qquad t \to 0^{+}. \tag{2.9}$$

Here the C_j, $j = 1, 2, 3, 4, 5$ *are positive and independent of* t *and* n.

(b) *In particular, if* ψ *satisfies*

$$\int_0^{C_6 t} \frac{\psi(\tau)}{\tau}\, d\tau = O(\psi(t)), \qquad t \to 0^{+}$$

then there exist $C_j > 0$, $j = 7, 8$ *independent of* t *and* n *such that*

$$E_n[f]_{W,p} = O\left(\psi\left(C_7 \frac{a_n}{n}\right)\right), \qquad n \to \infty \tag{2.10}$$

and

$$\omega_{r,p}(f, W, t) = O(\psi(C_8 t)), \qquad t \to 0^+. \tag{2.11}$$

We deduce the following analogue of Theorem 2.5.

THEOREM 2.7 (Characterization Theorem). *Let* $W \in \mathscr{E}$, $0 < \alpha \leqslant r$, $1 \leqslant p \leqslant \infty$, $f \in L_{p,W}(\mathbb{R})$ *and assume* (1.10).

(a) *Then the following are equivalent.*

$$\omega_{r,p}(f, W, t) = O(t^\alpha), \qquad t \to 0^+. \tag{2.12}$$

$$\|P_n^{*(r)} \Phi_{a_n/n}^r W\|_{L_p(\mathbb{R})} = O\left(\frac{n}{a_n}\right)^{r-\alpha}, \qquad n \to \infty. \tag{2.13}$$

(b) *In particular, the following are equivalent.*

$$\omega_{r,p}(f, W, t) = O(t^r), \qquad t \to 0^+. \tag{2.14}$$

$$\|P_n^{*(r)} \Phi_{a_n/n}^r W\|_{L_p(\mathbb{R})} = O(1), \qquad n \to \infty. \tag{2.15}$$

Remark 2.8. (a) We believe that is unlikely that (2.5) and (2.6) should hold with $\alpha = r$. Indeed it seems that the characterization (2.15) is the better replacement. We deduce that in the range for which $\omega_{r,p}(f, W, ;)$ and $\omega_{r+1,p}(f, W, ;)$ have different behaviors, $E_n[f]_{W,p}$ yields information on $\omega_{j,p}(f, W, ;)$ and $\|P_n^{*(j)} \Phi_{a_n/n}^j W\|_{L_p(\mathbb{R})}$ yields information on $\omega_{j,p}(f, W, ;)$ for $j = r$ and $j = r + 1$.

(b) Concerning the relationship between $\omega_{r,p}(f, W, ;)$ and $\omega_{r+1,p}(f, W, ;)$ we proved a Marchaud inequality in [2].

We now establish

THEOREM 2.9 (Quasi r-Monotonicity of the Modulus). *Let* $W \in \mathscr{E}$, $0 < p \leqslant \infty$, $f \in L_{p,W}(\mathbb{R})$, $t \in (0, D)$, $r \geqslant 1$, *and assume* (1.10). *Then there exists* $C_1 > 0$ *independent of* f *and* t *such that*

$$\omega_{r+1,p}(f, W, t) \leqslant C\omega_{r,p}(f, W, t). \tag{2.16}$$

2.3. *Estimates and Existence of* $f^{(k)}$, $k \geqslant 1$

We are able to prove the following existence theorem.

THEOREM 2.10. *Let* $W \in \mathscr{E}$, $0 < p \leqslant \infty$, $f \in L_{p,W}(\mathbb{R})$, $n \geqslant n_0$ *and* $q = \min(1, p)$. *Moreover assume* (1.10). *Then if*

$$\sum_{j=1}^{\infty} \left(\frac{2^{j-1}n}{a_{2^{j-1}n}}\right)^{kq} 2^{j\varepsilon} E_{2^{j-1}n}[f]_{W,p}^{q} < \infty$$

for some $\varepsilon > 0$ *and positive integer* k,

$$f^{(k)} W \in L_p(\mathbb{R})$$

and

$$\|(f - P_n^*)^{(k)} \Phi_{a_n/n}^{k} W\|_{L_p(\mathbb{R})} \leqslant C_1 \left(\sum_{j=1}^{\infty} \left(\frac{2^{j-1}n}{a_{2^{j-1}n}}\right)^{kq} 2^{j\varepsilon} E_{2^{j-1}n}[f]_{W,p}^{q}\right)^{1/q}. \quad (2.17)$$

Remark 2.11. It is possible under our hypotheses to reformulate all our results for $n \geqslant r$.

3. THE PROOFS OF THEOREMS 2.1 AND 2.3

In this section, we present the proofs of Theorems 2.1 and 2.3. To this end, we require three lemmas. Our first lemma concerns the functions a_u, σ, Φ_t, and $\Psi_{y,z}$.

LEMMA 3.1. *Let* $W \in \mathscr{E}$. *Then*

(a) *Given fixed* $\alpha > 1$, *we have uniformly for* $u > u_0$,

$$\left|\frac{a_{\alpha u}}{a_u} - 1\right| \sim T(a_u)^{-1}. \quad (3.1)$$

(b) *Given* $\alpha > 0$ *and* $\gamma > 1$ *we have uniformly for* $u \geqslant u_0$

(i)

$$Q(a_u) \sim u T(a_u)^{-1/2}, \quad (3.2)$$

(ii)

$$T(a_u) \sim T(a_{\alpha u}), \quad (3.3)$$

(iii)

$$\frac{Q(a_{\gamma u})}{Q(a_u)} > 1. \quad (3.4)$$

(c) *There exists* s_0 *and* v_0 *such that for* $s \in (0, s_0)$ *and* $v \geqslant v_0$, *we may write* $s = a_v/v$ *where* $v \geqslant v_0$. *Moreover,*

$$\sigma(s) = \sigma\left(\frac{a_v}{v}\right) = a_{\beta(v)} \tag{3.5}$$

where

$$v(1-\varepsilon) \leqslant \beta(v) \leqslant v.$$

(d) *Let* $a > 1$. *Then there exists* $C_1 > 0$ *such that for* $t/a \leqslant s \leqslant t$ *and* $0 < t \leqslant D$

$$1 \leqslant \frac{\sigma(s)}{\sigma(t)} \leqslant 1 + \frac{C_1}{T(\sigma(s))}. \tag{3.6}$$

Moreover, uniformly for s, t *above and* $x \in \mathbb{R}$

$$\Phi_s(x) \sim \Phi_t(x). \tag{3.7}$$

(e) *Given* $0 \leqslant s \leqslant t \leqslant D$, *there exists* $C > 0$ *independent of* s *and* t *such that*

$$T(\sigma(t))\left(1 - \frac{\sigma(t)}{\sigma(s)}\right) \leqslant C \log\left(2 + \frac{t}{s}\right). \tag{3.8}$$

(f) *Given* $u \leqslant v \leqslant u_0$ *for some large enough but fixed* u_0, *there exists positive constants* C_j, $j = 1, 2$ *independent of* u *and* v *such that*

$$(u/v)^{C_1 T(v)} \leqslant \frac{Q(u)}{Q(v)} \leqslant (u/v)^{C_2 T(u)}. \tag{3.9}$$

Proof. Part (a) is Lemma 2.2(d) in [4] while (3.2) is Lemma 2.2(b) in [4]. (3.3) is (2.2) of [1] and (3.4) is (2.9) of [4]. (3.5) is Lemma 3.1(a) of [4] and (3.6) is (2 14) of [1]. (3.7) is (2.18) of [1], (3.8) is (7.1) of [4], and (3.9) is (2.1) of [4]. ∎

Our next lemma is an estimate of the function $\Phi_{y,z}$ defined by (2.2).

LEMMA 3.2. *Let* $W \in \mathscr{E}$, $\varepsilon, \alpha > 0$. *Then there exists positive* C_j, $j = 1, 2$, *independent of* s, t, *and* x *such that for* $0 < s \leqslant t \leqslant D$,

$$C_1\left(\log\left(2 + \frac{t}{s}\right)\right)^{-\alpha/2} \leqslant (\sup_{x \in \mathbb{R}} (\Psi_{t,s}(x))^{\alpha} \leqslant C_2\left(\frac{t}{s}\right)^{\varepsilon}. \tag{3.10}$$

Proof. The lower bound in (3.10) was established in (7.2) of [4]. Thus it suffices to establish the corresponding upper bound. Firstly if $|x| \leqslant \sigma(t)$, then the result follows by (3.5) of [4] since in this case

$$\Psi_{t,s}(x) \leqslant C_1$$

for some positive constant C_1 independent of s, t, and x. Thus we may assume without loss of generality that $|x| > \sigma(t)$. We first claim that

$$\Phi_t(x) \leqslant C_2 \left|1 - \frac{|x|}{\sigma(2t)}\right|^{1/2}$$

for some positive constant C_2 independent of x and t.

To see this, first observe that (3.6) implies that

$$\left|1 - \frac{|x|}{\sigma(2t)}\right|^{1/2} \geqslant C_3 \max\left(\left|1 - \frac{|x|}{\sigma(t)}\right|^{1/2}, T(\sigma(t))^{-1/2}\right)$$

for our range of $|x|$. Then using the estimate above yields

$$\Phi_t(x) \leqslant 2/C_3 \left|1 - \frac{|x|}{\sigma(2t)}\right|^{1/2}.$$

Now using the estimate above, the triangle inequality and the definition of Φ_s, we have

$$\begin{aligned}
\Phi_t(x) &\leqslant \left|1 - \frac{|x|}{\sigma(s)}\right|^{1/2} + \left|1 - \frac{\sigma(s)}{\sigma(2t)}\right|^{1/2} \left[\left|1 - \frac{|x|}{\sigma(s)}\right|^{1/2} + 1\right] \\
&\leqslant C_4 \left[\Phi_s(x) + \left(\frac{\sigma(s)}{\sigma(2t)}\right)^{1/2} \left|1 - \frac{\sigma(2t)}{\sigma(s)}\right|^{1/2} \Phi_s(x)\right] \\
&\quad + C_4 \left[\left(\frac{\sigma(s)}{\sigma(t)}\right)^{1/2} \left|1 - \frac{\sigma(2t)}{\sigma(s)}\right|^{1/2} T(\sigma(2t))^{1/2} \left(\frac{T(\sigma(s))}{T(\sigma(2t))}\right)^{1/2} \Phi_s(x)\right] \\
&\leqslant C_5 \left(\frac{T(\sigma(s))}{T(\sigma(t))}\right)^{1/2} \left(\frac{\sigma(s)}{\sigma(t)}\right)^{1/2} \sqrt{\log\left(2 + \frac{2t}{s}\right)}\, \Phi_s(x),
\end{aligned}$$

where in the last line we used (3.8). We observe that the positive constant C_5 is independent of t, s, and x.

We now estimate each of the terms in (3.11). Thus let $\varepsilon > 0$ be given. By Lemma 3.1(c), we may write $s = a_u/u$ and $2t = a_v/v$ where $u \geqslant v \geqslant v_0$ and v_0 is a large enough but fixed real number. Observe that

$$a_{\beta(u)} = \sigma(s) \geqslant \sigma(2t) = a_{\beta_v}$$

with $\beta(u) \geqslant \beta(v)$, $\beta(u) = u(1 + o(1))$ and $\beta(v) = v(1 + o(1))$.

Then as T is quasi-increasing it follows from (3.2), (3.3), (3.4), and (3.9) that

$$(u/v) \leqslant C_6(t/s)^{1/1-\varepsilon}. \tag{3.12}$$

Now applying (1.1) with $y = \sigma(s)$ and $x = \sigma(2t)$ together with (3.2) and (3.12) then yields

$$\left(\frac{T(\sigma(s))}{T(\sigma(t))}\right)^{1/2} \leqslant C_7(t/s)^{\varepsilon}$$

and

$$\left(\frac{\sigma(s)}{\sigma(t)}\right)^{1/2} \leqslant C_8(t/s)^{\varepsilon}.$$

Inserting these estimates into (3.11), recalling that logarithms grow slower than any polynomial and dividing by $\Phi_s(x)$ yields the upper bound in (3.10) and hence the lemma. ∎

Our final lemma concerns (1.13) and an extension of the Markov–Bernstein inequality (1.10).

LEMMA 3.3. *Let $W \in \mathscr{E}$, $r \geqslant 1$, $0 < p \leqslant \infty$, $f \in L_{p,W}(\mathbb{R})$ and assume* (1.10).

(a) *Then if $n \geqslant N_0$ and $P \in \mathscr{P}_n$, there exists $C_1 \neq C_1(n, P)$ such that*

$$\|P^{(r+1)}\Phi^{r+1}_{a_n/n} W\|_{L_p(\mathbb{R})} \leqslant C_1 \frac{n}{a_n} \|P^{(r)}\Phi^{r}_{a_n/n} W\|_{L_p(\mathbb{R})}. \tag{3.13}$$

(b) *Let $0 < t < D$ and define $n(t)$ by* (1.13). *Then uniformly for f, t, and $\lambda \in [1, D/t]$,*

$$\frac{a_{n(\lambda t)}}{n(\lambda t)} \leqslant \lambda t < 2\frac{a_{n(\lambda t)}}{n(\lambda t)}, \tag{3.14}$$

$$K_{r,p}(f, W, (\lambda t)^r) \sim K_{r,p}\left(f, W, \left(\frac{a_{n(\lambda t)}}{n(\lambda t)}\right)^r\right), \tag{3.15}$$

and

$$\omega_{r,p}(f, W, \lambda t) \sim \omega_{r,p}\left(f, W, \frac{a_{n(\lambda t)}}{n(\lambda t)}\right). \tag{3.16}$$

Proof. Part (a) appeared first in [1, Lemma 3.1]. Part (b) for $\lambda = 1$, follows from [1, (2.25)], [1, (1.23)], and [1, (1.14)]. The general case follows by replacing t by λt and using (1.15), (1.16), and (3.7). ∎

We are ready for the proofs of Theorem 2.1 and 2.3.
We begin with

The Proof of Theorem 2.1. Let $t \in (0, D)$, $\lambda \in [1, D/t]$, $\varepsilon > 0$ and determine $n(t)$ and $n(\lambda t)$ by (1.13). By (1.12) we may choose $P \in \mathscr{P}_{n(t)}$ such that

$$\|(f - P)\, W\|_{L_p(\mathbb{R})} + t^r\, \|WP^{(r)}\Phi_t^r\|_{L_p(\mathbb{R})} \leqslant 2K_{r,p}(f, W, t^r). \tag{3.17}$$

Next by (1.11), (1.16), (1.18), and (3.16) we may choose $R \in \mathscr{P}_{n(\lambda t)}$ such that

$$\begin{aligned} \|(R - P)\, W\|_{L_p(\mathbb{R})} &\leqslant C_1 w_{r,p}\left(P, W, \frac{a_{n(\lambda t)}}{n(\lambda t)}\right) \\ &\leqslant C_2 w_{r,p}(P, W, \lambda t) \leqslant C_3 (\lambda t)^r\, \|P^{(r)} W \Phi_{\lambda t}^r\|_{L_p(\mathbb{R})}, \end{aligned} \tag{3.18}$$

where $C_3 \neq C_3(f, t, \lambda)$.
Similarly we obtain

$$\begin{aligned} (\lambda t)^r\, \|WR^{(r)}\Phi_{\lambda t}^r\|_{L_p(\mathbb{R})} &\leqslant C_4 K_{r,p}(P, W, (\lambda t)^r) \leqslant C_5 w_{r,p}(P, W, \lambda t) \\ &\leqslant C_6 (\lambda t)^r\, \|P^{(r)} W \Phi_{\lambda t}^r\|_{L_p(\mathbb{R})} \end{aligned} \tag{3.19}$$

for some $C_6 \neq C_6(f, t, \lambda)$.
Let $q = \min(1, p)$. Then (1.12), (3.17), (3.18), and (3.19) yield

$$\begin{aligned} K_{r,p}(f, W, (\lambda t)^r)^q &\leqslant C_7 (\|(f - R)\, W\|_{L_p(\mathbb{R})}^q + (\lambda t)^{rq}\, \|R^{(r)} W \Phi_{\lambda t}^r\|_{L_p(\mathbb{R})}^q) \\ &\leqslant C_8 (\|(f - P)\, W\|_{L_p(\mathbb{R})}^q + (\lambda t)^{rq}\, \|P^{(r)} W \Phi_{\lambda t}^r\|_{L_p(\mathbb{R})}^q) \\ &\leqslant C_9 \lambda^{rq} (\sup_{x \in \mathbb{R}} \Psi_{\lambda t, t}(x))^{rq}\, K_{r,p}(f, W, t^r). \end{aligned}$$

Here $C_9 \neq C_9(f, t, \lambda)$.
Taking qth roots and using (1.14) gives (2.1). (2.3) then follows using (3.10). ∎

With Theorem 2.1 at our disposal, we may proceed with

The Proof of Theorem 2.3. Our method of proof uses an idea from [8]. Choose $t_0 \in [t, D]$. We first show that (2.4) implies that

$$\omega_{r,p}(f, W, t_0) = 0. \tag{3.20}$$

This follows as given $\varepsilon > 0$, we have by Theorem 2.1 that uniformly for $t \in (0, D)$,

$$\omega_{r,p}(f, W, t_0) = \omega_{r,p}\left(f, W, \frac{t_0 t}{t}\right) \leqslant C_1 \frac{\omega_{r,p}(f, W, t)}{t^{r+\varepsilon}},$$

where $C_1 \neq C_1(f, t)$.

We see now why it is crucial that (2.3) should hold uniformly for $\lambda \in [1, D/t]$.

Then (2.4) implies (3.20) and so (1.14) implies

$$K_{r,p}(f, W, t_0^r) = 0. \tag{3.21}$$

Here $n = n(t_0)$ is defined by (1.13). By (3.21), we may choose a sequence of polynomials $(P_i)_{i=1}^{\infty} \in \mathscr{P}_n$ such that

$$\|(f - P_i)\, W\|_{L_p(\mathbb{R})} + t_0^r\, \|P_i^{(r)} \Phi^r_{a_n/n} W\|_{L_p(\mathbb{R})} \leqslant 2^{-i} t_0^r. \tag{3.22}$$

Then for a.e. $x \in \mathbb{R}$ we have,

$$f(x) = P_i(x) + \sum_{j=i}^{\infty} (P_{j+1} - P_j)(x)$$

and so (3.21) and (3.22) give

$$\|f^{(r)} \Phi^r_{a_n/n} W\|_{L_p(\mathbb{R})} \leqslant C_1 \left(2^{-i} + \sum_{j=i}^{\infty} 2^{-(j+1)} + 2^{-j}\right) \leqslant C_2 2^{-i}. \tag{3.23}$$

As (3.23) holds for each $i \geqslant 1$, we must have

$$\|f^{(r)} \Phi^r_{a_n/n} W\|_{L_p(\mathbb{R})} = 0,$$

which implies that for a.e. $x \in \mathbb{R}$

$$f^{(r)} \Phi^r_{a_n/n} W(x) = 0$$

or f is a polynomial of degree $r - 1$ a.e. ∎

4. OUR REMAINING PROOFS

In this section, we present the proofs of Theorems 2.6, 2.7, 2.9, and 2.10 following ideas from [5] and [8].

4.1. *Characterization Theorem*

We begin with

The Proof of Theorem 2.6. Let $P_n^*(P_{2n}^*)$ be the best approximant to P_{2n}^* from $\mathscr{P}_n$ satisfying,

$$\|(P_{2n}^* - P_n^*(P_{2n}^*))\, W\|_{L_p(\mathbb{R})} = E_n[P_{2n}^*]_{W,\,p}. \tag{4.1}$$

Then using (1.4),

$$I_n^q := \|(P_{2n}^* - P_n^*(P_{2n}^*))\, W\|_{L_p(\mathbb{R})} \geqslant C(E_n[f]_{W,\,p} - E_{2n}[f]_{W,\,p}) \tag{4.2}$$

for some $C \neq C(n, f)$.

Also, by (1.11), (1.15), (1.18), (2.7), and (3.1),

$$I_n \leqslant C_1 \omega_{r,\,p}\left(P_{2n}^*,\, W,\, \frac{a_n}{n}\right) \leqslant C_2 \psi\left(\frac{a_{2n}}{2n}\right). \tag{4.3}$$

Here, $C_2 \neq C_2(n)$.

Then (4.2) and (4.3) give

$$E_n[f]_{W,\,p} \leqslant C_3 \sum_{k=0}^{\infty} I_{2^k n} \leqslant C_4 \sum_{k=1}^{\infty} \psi\left(\frac{a_{2^k n}}{2^k n}\right) = C_4 S_n, \tag{4.4}$$

where

$$S_n := \sum_{k=1}^{\infty} \psi\left(\frac{a_{2^k n}}{2^k n}\right), \qquad n \geqslant 1 \tag{4.5}$$

and $C_4 \neq C_4(n)$.

We now estimate (4.5) in terms of an integral.

First observe using (3.1), that there exists n_0 such that uniformly for $k \geqslant 1$ and $n \geqslant n_0$,

$$\int_{a_{2^k n}/2^k n}^{a_{2^{k-1} n}/2^{k-1} n} \frac{1}{\tau}\, d\tau \geqslant \frac{1}{2} \log 2.$$

Then the quasi-monotonicity of ψ gives

$$S_n \leqslant C_5 \sum_{k=1}^{\infty} \int_{a_{2^k n}/2^k n}^{a_{2^{k-1}n}/2^{k-1}n} \frac{\psi(\tau)}{\tau}\, d\tau, \tag{4.6}$$

where $C_6 \neq C_6(n)$.

Substituting (4.6) into (4.4) gives (2.8).

Now let $0<t<D$ and define $n := n(t)$ by (1.13).

Then using (1.4), (1.14), (1.16), (2.7), (3.1), and (4.4), we proceed much as in the proof of (2.8) and obtain

$$\begin{aligned}
\omega_{r,p}(f, W, t) &\leqslant C_1 \omega_{r,p}\left(f, W, \frac{a_{2n}}{2n}\right) \\
&\leqslant C_2 K_{r,p}\left(f, W, \left(\frac{a_{2n}}{2n}\right)^r\right) \\
&\leqslant C_3 \left(\|(f-P^*_{2n})\, W\|_{L_p(\mathbb{R})} + \left(\frac{a_{2n}}{2n}\right)^r \|P^{*(r)}_{2n} \Phi^r_{a_{2n}/2n} W\|_{L_p(\mathbb{R})}\right) \\
&\leqslant C_4 \left(E_{2n}[f]_{W,p} + \psi\left(\frac{a_{2n}}{2n}\right)\right) \\
&\leqslant C_5 \left(\sum_{k=0}^{\infty} \psi\left(\frac{a_{2^{k+1}n}}{2^{k+1}n}\right)\right) \leqslant C_6 \int_0^{C_7 t} \frac{\psi(\tau)}{\tau}\, d\tau.
\end{aligned} \tag{4.7}$$

Here $C_6 \neq C_6(t)$. Thus we have (2.9), (2.10) and (2.11) then follow easily. ∎

We may proceed with

The Proof of Theorem 2.7. We apply Theorem 2.6 with $\psi(\tau) := \tau^\alpha$. This then shows that (2.13) implies (2.12). The other way follows from (1.14) and (1.16). The equivalence of (2.14) and (2.15) follow from part (a) of Theorem 2.7 by setting $\alpha = r$. ∎

4.2. *Existence Theorems and Monotonicity*

In this section, we present the proofs of Theorems 2.9 and 2.10.

We begin with

The Proof of Theorem 2.9. Let $q = \min(1, p)$ and let P^*_n be the best approximant to f satisfying (1.17). Then (1.11), (1.12), (1.14), (1.16), and (3.13) give for $n \geqslant n_0$,

$$\omega_{r+1,p}\left(f, W, \frac{a_n}{n}\right)^q$$
$$\leqslant C_1\left(\|(f-P_n^*)\,W\|^q_{L_p(\mathbb{R})}+\left(\frac{a_n}{n}\right)^{(r+1)q}\|P_n^{*(r+1)}\Phi^{r+1}_{a_n/n}W\|^q_{L_p(\mathbb{R})}\right)$$
$$\leqslant C_2\left(E_n[f]^q_{W,p}+\left(\frac{a_n}{n}\right)^{rq}\|P_n^{*(r)}\Phi^{r}_{a_n/n}W\|^q_{L_p(\mathbb{R})}\right)$$
$$\leqslant C_3\omega_{r,p}\left(f, W, \frac{a_n}{n}\right)^q.$$

Here $C_3 \neq C_3(f, n)$.

Now let $0<t<D$ and determine $n:=n(t)$ by (1.13). Then (3.16) with $\lambda=1$ and (4.8) together imply (2.16). ∎

We finish this section with

The Proof of Theorem 2.10. Let P_n^* be the best approximant to f satisfying (1.17). Then much as in the proof of Theorem 2.3, we write for a.e. $x\in\mathbb{R}$,

$$f(x)=P_n^*(x)+\sum_{j=1}^{\infty}\left(P^*_{2^jn}(x)-P^*_{2^{j-1}n}(x)\right). \tag{4.9}$$

Now let $\varepsilon>0$ and apply (4.9) together with (3.13), (3.10) and ε/q. This gives,

$$\|(f-P_n^*)^{(k)}\,\Phi^k_{a_n/n}W\|^q_{L_p(\mathbb{R})}\leqslant C_1\sum_{j=1}^{\infty}2^{j\varepsilon}\left(\frac{2^jn}{a_{2^jn}}\right)^{kq}\|(P^*_{2^jn}-P^*_{2^{j-1}n})\,W\|^q_{L^p(\mathbb{R})}$$
$$\leqslant C_2\sum_{j=1}^{\infty}\left(\frac{2^{j-1}n}{a_{2^{j-1}n}}\right)^{kq}2^{j\varepsilon}E^q_{2^{j-1}n}[f]_{W,p}.$$

Here, $C_2\neq C_2(n, f)$. Taking qth roots gives the theorem. ∎

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